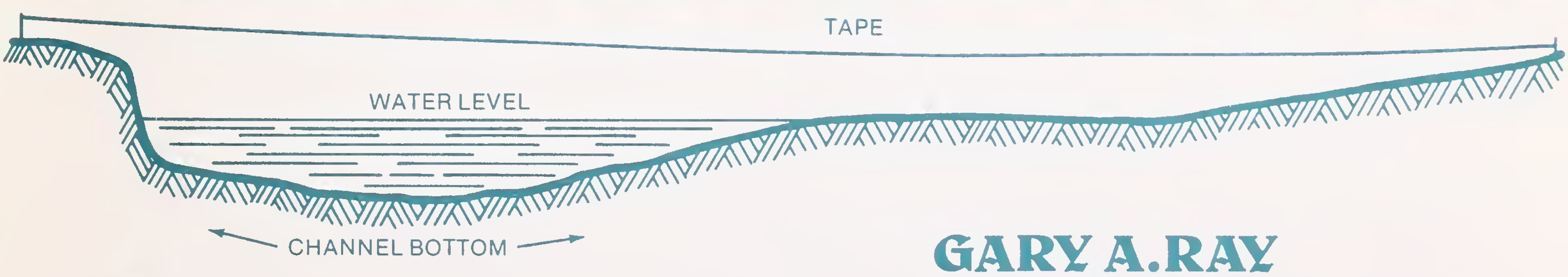
Measuring cross sections using a sag tape: A generalized procedure



GARY A. RAY
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RESEARCH SUMMARY

A procedure was developed for surveying cross sections using a sag tape with unequal end elevations. Information needed to perform the necessary calculations includes: the tape weight (lb) per foot of length; the difference in elevation (ft) between the two ends of the tape; the tension (lb) on the tape; and the tape length (ft). The procedure is as accurate as traditional engineer's level surveys and is faster and easier. Examples of a typical field survey and the resulting cross section plot are provided. The procedure is easily programed for a digital computer; a flow diagram for calculating and plotting cross sections is provided.

INTRODUCTION

Hydrologists, geologists, and civil engineers commonly use cross section surveys consisting of a series of paired horizontal and vertical measurements to describe the shape, or changes in shape, of the earth's surface. Usually, cross section surveys are made by measuring horizontal distances along a tape stretched between two reference stakes and by measuring vertical distances with an engineer's level and a level rod. Although this technique can be fairly accurate, it is time consuming, requires two people, and has a relatively large possibility for observer error.

A simpler procedure is to measure vertical distances with a level rod directly from the tape to the point of interest on the underlying surface. However, this procedure requires that a correction be made to account for the inevitable sag in the tape. The calculation is relatively simple if both ends of the tape are at the same elevation so that the low point of the tape is located in the center of the cross section. Computer programs to calculate (DEBRIS) and plot (PLOT D) cross sections taken with a sag tape with equal end tape elevations are presently available (USDA Forest Service, Watershed Systems Development Unit 1975). The programs were originally designed to evaluate sediment accumulations in debris basins where it is relatively easy to set the ends of the measuring tape at the same elevation. Unfortunately, it has been our experience that the programs have limited application because it is usually difficult and often impossible to establish the ends of the tape at the same elevation.

To solve this problem, we have developed a procedure which permits the use of the sag tape for any combination of end elevations. This procedure can be used in a much wider variety of applications than the previous method. For example, hydrologists can use it for documenting channel bank erosion and bottom aggradation and degradation; for evaluating aquatic habitat conditions; for establishing water surface profiles; and for determining the amount of erosion and deposition on hill slopes and construction areas.

CALCULATION OF SAG CORRECTION

A tape, suspended between two end points, describes a catenary curve. If the end points of the tape are at the same elevation and if the weight per foot of the tape is known and the tension in the tape is measured, it is easy to calculate the shape of the catenary curve according to the equation given by Thomas (1960). Once this shape is known, you can make the necessary corrections in vertical distance and you can correct distances measured along the tape to true horizontal distances thereby adding to the accuracy of the survey.

Marks (1951) provides a method for the solution of a catenary curve with unequal end elevations. Unfortunately, the procedure given is inadequate for two reasons: (1) it assumes that the true x coordinate distance between the ends of the tape (the value

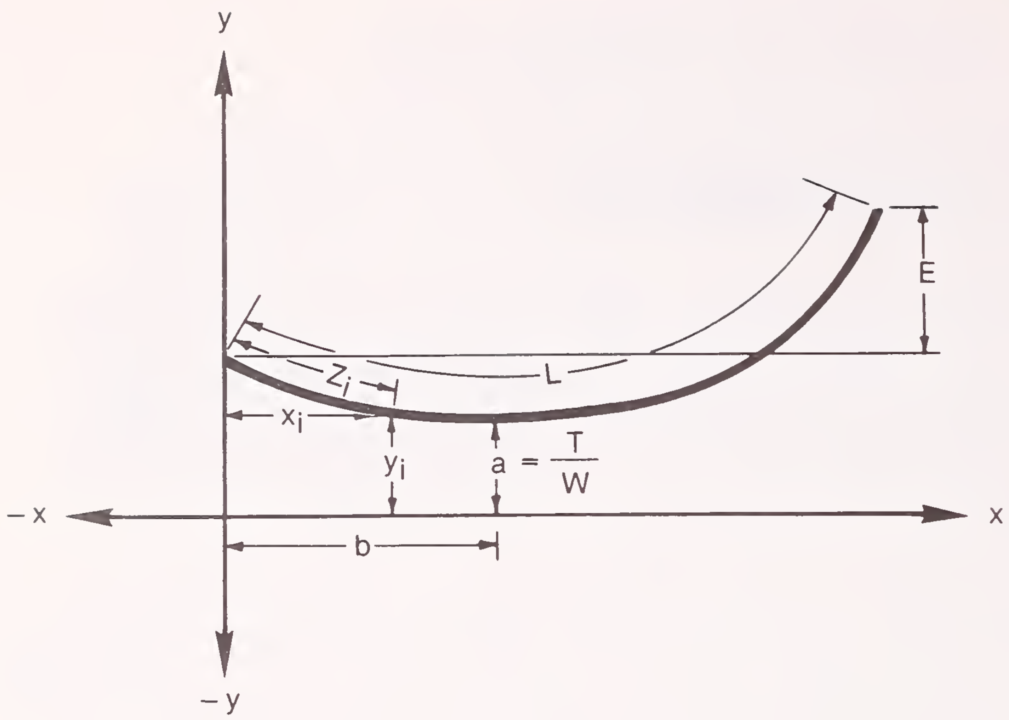


Figure 1.--Dimensions required to calculate the shape of a catenary curve with unequal end elevations.

c in our appendix derivation) is known; and (2) the formulae required to derive the tables are not given, thereby making the procedure impossible to accurately implement. In practical application, the true x coordinate distance between the ends of the tape is unknown, and formulae must be available for efficient solution on a computer.

In order to describe the catenary curve defined by a tape suspended between two points of unequal elevation, it is necessary to locate the x coordinate of the low point of the curve (defined as b, fig. 1). This is possible if the tape length (L) in feet, the tape tension (T) in pounds, the tape weight per unit length (W) in pounds per foot, and the elevations of the tape ends in feet are known. The elevation of the right tape end minus the elevation of the left tape end defines the value of E in the following calculations.

The value of b is computed as follows:

$$b = a \ln\left[\{(L-E)/a\}\{1/(1-1/K)\}\right]$$

where:

$$a = T/W$$

$$K = \{-H+(H^2-4)^{0.5}\}/2.$$

$$H = -\{(L^2-E^2)/a^2\}-2.$$

When b is known, the x coordinate of any point on the catenary (x_i) can be calculated for a given tape distance z_i with the following:

$$x_i = a \sinh^{-1}\left(\frac{z_i}{a} + \sinh\frac{-b}{a}\right) + b$$

and the accompanying y coordinate (y_i) calculated as:

$$y_i = a \cosh\{(x_i-b)/a\}.$$

Derivations of these equations are given in appendix 1.

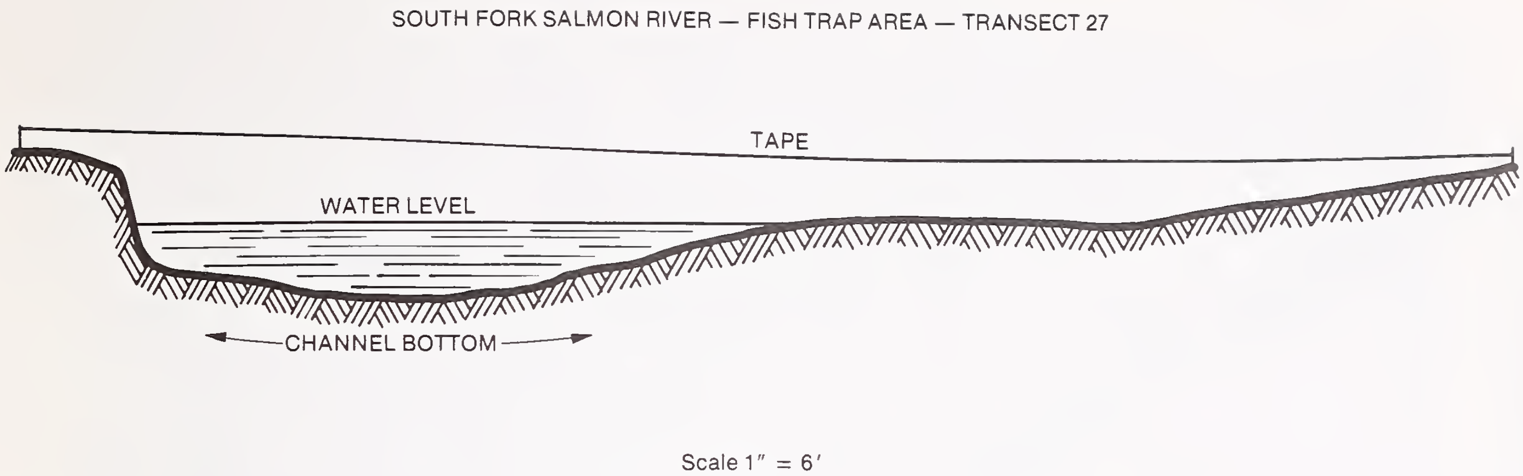


Figure 2.--Example of an actual stream channel cross section plot using the sag tape procedure, showing the channel bottom and the water level.

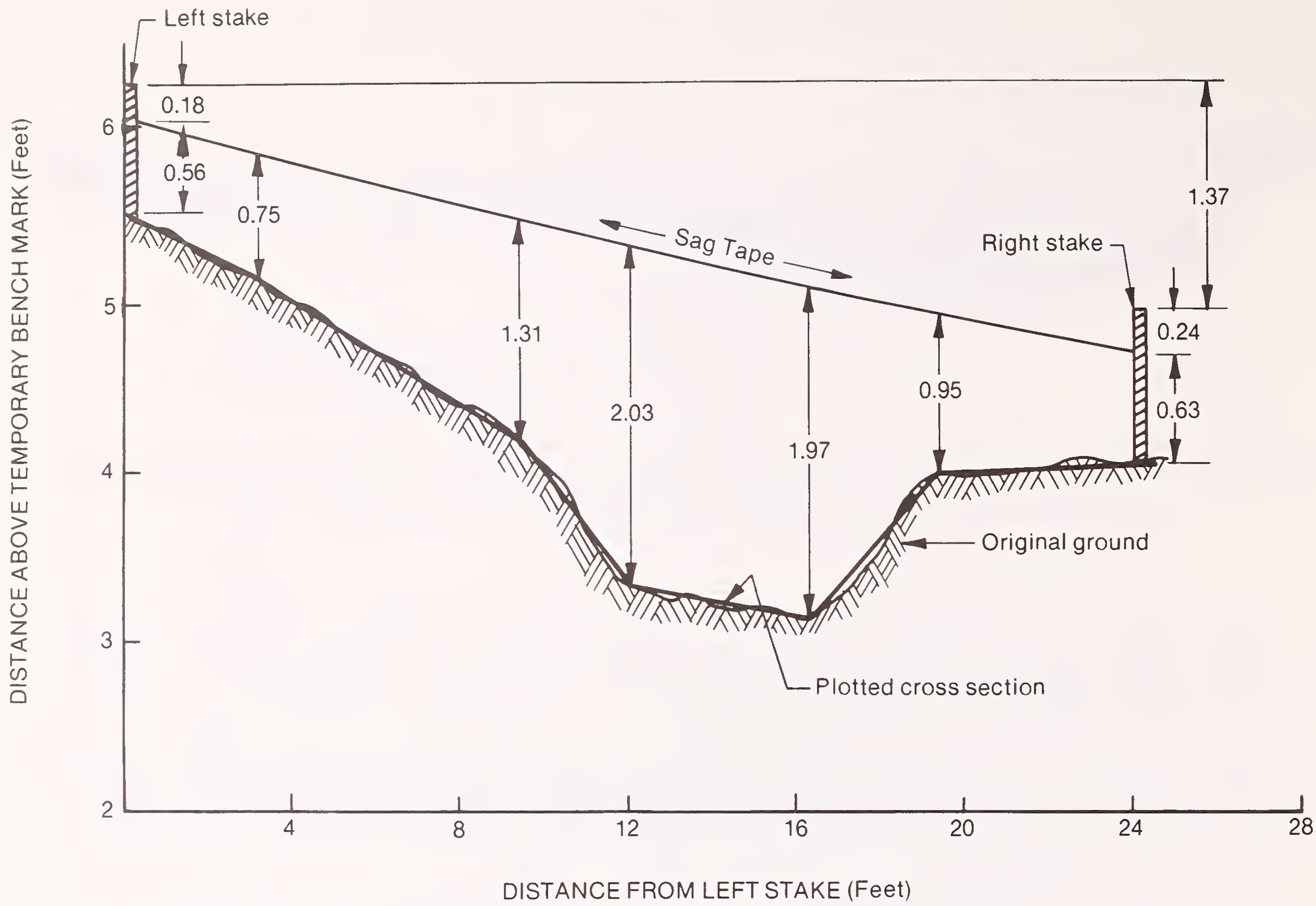
For very precise surveys, corrections can be made in the above equation to account for tape stretch, heat expansion, and the additional tension provided by the weight of the tape. Such corrections are not warranted for most field applications. Interested readers should refer to standard references (e.g., Marks 1951) to apply these corrections.

When the x and y coordinates of the tape for any tape reading are known, the y coordinate of the surface being surveyed is obtained by subtracting the measured vertical distance between the tape and the surface being surveyed from the y coordinate of the tape.

The above calculations are easily programed for computer analysis. We have developed a program for a Hewlett Packard 9821A programable calculator equipped with 423 registers that provides a plot of the cross section and the survey tape (fig. 2). A flow chart for the program is given in appendix 2. Additional field data collected at the time of the cross section survey can also be incorporated into the cross section analysis. For example, water depth data collected while surveying channel cross sections make it possible to plot the water level in figure 2.

FIELD PROCEDURES

Stable end stakes must be installed at the ends of each cross section prior to making a cross section survey. Many times, it is necessary to remeasure cross sections periodically to document changes over time. In this case, end stakes should be installed permanently to insure stability. An accurate survey is required to establish the difference in elevation between the two end stakes. An engineer's level and level rod are suggested for this purpose. Once the relative elevations of the stakes are known, no additional surveys are needed unless there is suspicion that the stakes may have moved.



Field Notes

Date 6/12/77, Location South fork Salmon River
Cross Section No. 24, Observer(s) L. Jones
Tape wgt./ft. 0.012714 (lb/ft) Tape Tension 16.5 (lb)
Elevation[1] right top of stake 4.98 (ft.)
Elevation left top of stake 6.35 (ft.)
Top right stake to tape 0.24 (ft.)
Top left stake to tape 0.18 (ft.)

Tape distance	Vertical
0.0	0.56
3.2	0.75
9.5	1.31
12.1	2.03
16.3	1.97
19.3	0.95
24.2	0.63

[1]Based on a temporary bench mark elevation of 10.00 feet.

Figure 3.--Example of field notes and the resulting cross section plot.

The actual cross section survey requires a steel tape, a spring scale graduated up to about 30 pounds, a tape clamp, and a level rod. All equipment can be obtained from survey equipment suppliers. The zero end of the tape is attached to the left[1/] end stake and the tape is stretched to the right end stake where it is attached to the spring scale with the use of a tape clamp. Care must be taken to assure that the tape is not touching anything for its entire length. Tension of at least 5 pounds plus 1 pound for each 10 feet in length is applied to the tape. Tape tension should be increased when conditions are windy. Excessive tape sway may make it necessary to delay field operations when using long tape spans (>100 feet) under high wind conditions (>15-20 miles per hour), especially when winds are blowing normal to the tape.

Paired tape and vertical distances are measured to the desired precision along the tape, including the points of attachment of each end of the tape, in order to locate the tape with respect to the surface being surveyed. Measurements are taken as necessary in order to define the shape of the cross section. Straight lines are assumed between measurement points on the surface being surveyed, so it is desirable to take readings at all major breaks in slope along the cross section. The distance from the top of each stake to the end of the tape must be measured in order to reference the height of the ends of the tape. An example of field notes for a cross section survey and of the resulting cross section plot is shown in figure 3.

Calculations from the survey data are as follows:

L = length of tape = 24.2 ft

$$\begin{aligned} E &= \text{elevation of right end of tape minus elevation of left end of tape} \\ &= (4.98 - 0.24) - (6.35 - 0.18) \\ &= -1.43 \text{ ft} \end{aligned}$$

$$\begin{aligned} a &= \text{tension/tape weight per ft} \\ &= 16.5 \text{ lb}/0.012714 \text{ lb/ft} \\ &= 1297.78197300 \text{ ft} \end{aligned}$$

$$\begin{aligned} H &= -\{(24.2^2 - (-1.43)^2)/a^2\} - 2 \\ &= -2.00034650 \end{aligned}$$

$$K = (-H + \sqrt{H^2 - 4})/2 = 1.01878867$$

$$\begin{aligned} b &= a \ln \left[\{(24.2 + 1.43)/a\}\{1/(1 - 1/K)\}\right] \\ &= 88.85524249. \end{aligned}$$

The x and y coordinates of the tape and the ground surface are calculated as in the following examples:

At $z_1 = 0$, $x_1 = 0$,

$$y_1 = a \cosh \left(\frac{-b}{a}\right) = 1300.82,$$

ground surface $= y_1 - 0.56 = 1300.26$.

[1/] The convention of whether to define right and left by looking upstream or downstream is the surveyor's choice; use should be consistant once the choice is made.

At $z_4 = 12.1$, $x_4 = a \sinh^{-1}(\frac{12.1}{a} + \sinh\frac{-b}{a}) + b = 12.08$,

$y_4 = a \cosh(\frac{12.08-b}{a}) = 1300.05$,

ground surface $= y_4 - 2.03 = 1298.02$.

At $z_7 = 24.2$, $x_7 = a \sinh^{-1}(\frac{24.2}{a}) + \sinh\frac{-b}{a}) + b = 24.16$,

$y_7 = a \cosh(\frac{24.16-b}{a}) = 1299.39$,

ground surface $= y_7 - 0.63 = 1298.76$.

Some peculiarities of the calculations require explanation. First, the values a, H, K, and b must be carried to at least 8 significant digits to assure precise calculation of the tape coordinates. Also, the value for a is calculated by dividing the tape tension by the tape weight per foot of length. A large number results (a = 1297.78197300 ft in the example) because the tape tension is large compared to the tape weight per foot of length. The value a is actually the distance between the low point of the tape and an imaginary horizontal coordinate plane (see fig. 1). All other positions on the tape (the calculated values for y_i) are referenced to the same coordinate plane and are likewise large. A constant can be added or subtracted from the computed values for y_i to reference the values to any other coordinate plane. For example, a constant of 1294.65 was subtracted from the y_i values computed above in order to reference the values used to plot the cross section in figure 3 to the 10.00 foot bench mark used for the field survey. The constant was determined by subtracting the elevation of the left end of the tape based on the temporary bench mark (6.17) from the computed value of y_i for the left end of the tape (1300.82).

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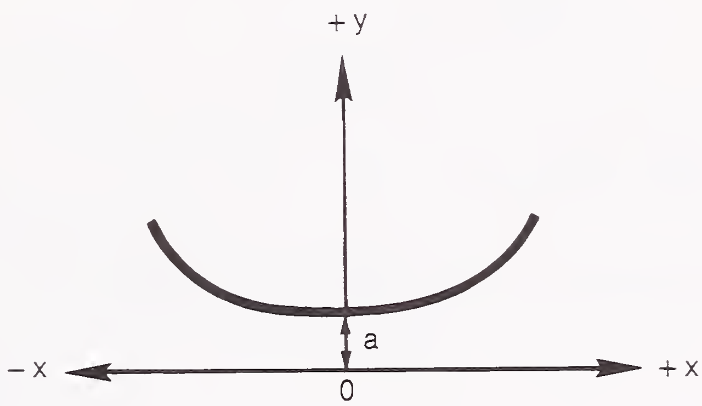
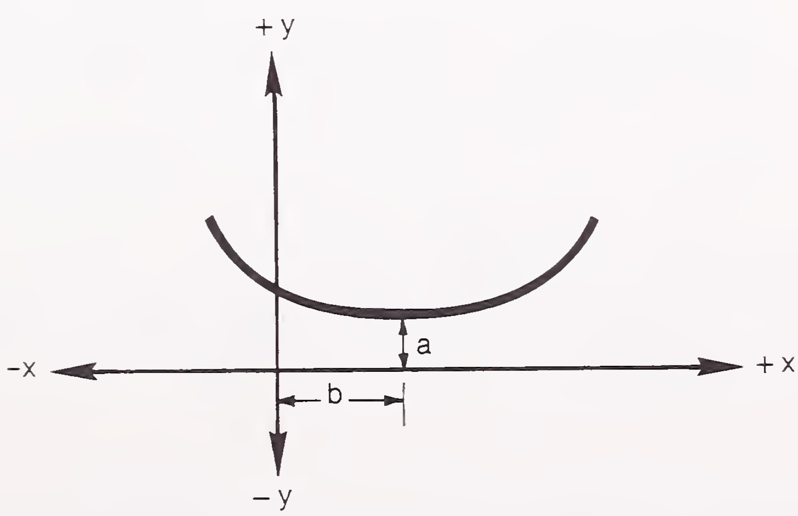
APPENDIX 1

Derivation of equations for calculating the position of a sag tape with unequal end elevations

The equation of a catenary (hanging chain) is $y = a \cosh \frac{x}{a}$ where a is tension weight per unit length, and the low point is at $x = 0$ (Thomas 1960).

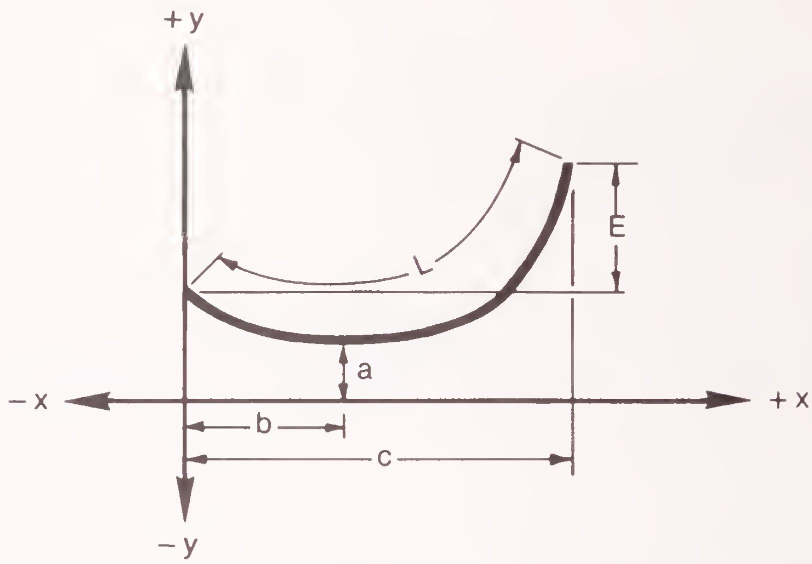


A more general equation, where the low point is at $x = b$, is $y = a \cosh \frac{x-b}{a}$.



Let one stake be at x = 0, the other stake at x = c. Let the length of the tape between the two stakes be L, and the difference in the height of the stakes be E. The problem now is: given a, E, and L, find b.

$$f(x) = a \cosh \frac{x-b}{a}$$



L is the arc length from 0 to c, so

$$L = \int_0^c \sqrt{1 + (f'(x))^2}\, dx = \int_0^c \sqrt{1 + \sinh^2 \frac{x-b}{a}}\, dx$$

$$= \int_0^c \cosh \frac{x-b}{a}\, dx = a \sinh \frac{c-b}{a} - a \sinh \frac{-b}{a}$$

so,

$$\sinh \frac{c-b}{a} - \sinh \frac{-b}{a} = \frac{L}{a}. \qquad (1)$$

Solving for f(x) at 0 and c,

$$E^{2/} = a \cosh \frac{c-b}{a} - a \cosh \frac{-b}{a}$$

so,

$$\cosh \frac{c-b}{a} - \cosh \frac{-b}{a} = \frac{E}{a}. \qquad (2)$$

2/ In this derivation, E is defined as the elevation of the *right* end of tape (f(x) at c) minus the elevation of the *left* end of the tape (f(x) at 0).

Using the definitions of cosh (x) and sinh (x) with $\frac{c-b}{a} = p$, $\frac{-b}{a} = q$, equations (1) and (2) become:

$$\frac{e^p - e^{-p}}{2} - \frac{e^q - e^{-q}}{2} = \frac{L}{a} \qquad (3)$$

and,

$$\frac{e^p + e^{-p}}{2} - \frac{e^q + e^{-q}}{2} = \frac{E}{a}\,. \qquad (4)$$

Adding (3) and (4),

$$e^p - e^q = \frac{L + E}{a}.$$

Subtracting (4) from (3),

$$-e^{-p} + e^{-q} = \frac{L - E}{a}$$

so,

$$e^{\frac{c}{a} - \frac{b}{a}} - e^{-\frac{b}{a}} = \frac{L + E}{a}$$

$$-e^{-\frac{c}{a} + \frac{b}{a}} + e^{\frac{b}{a}} = \frac{L - E}{a}$$

or,

$$-e^{-\frac{b}{a}} \left[e^{\frac{c}{a}} - 1 \right] = \frac{L + E}{a}$$

$$e^{\frac{b}{a}} \left[-e^{-\frac{c}{a}} + 1 \right] = \frac{L - E}{a} \qquad (5)$$

so,

$$-e^{-\frac{b}{a}} = \frac{a}{L - E} \left[-e^{-\frac{c}{a}} + 1 \right]. \qquad (6)$$

Substituting (6) into (5) gives:

$$\left[-e^{-\frac{c}{a}} + 1 \right] \left[e^{\frac{c}{a}} - 1 \right] = \frac{L^2 - E^2}{a^2}$$

or,

$$e^{\frac{c}{a}} + e^{-\frac{c}{a}} - 2 = \frac{L^2 - E^2}{a^2}\,.$$

Multiplying both sides by $e^{\frac{c}{a}}$ gives:

$$\left[e^{\frac{c}{a}}\right]^2 + \left(-\frac{L^2 - E^2}{a^2} - 2\right)\left(e^{\frac{c}{a}}\right) + 1 = 0. \tag{7}$$

Let $K = e^{\frac{c}{a}}$ and $H = -\frac{L^2 - E^2}{a^2} - 2$ and (7) becomes:

$$K^2 + HK + 1 = 0$$

so,

$$K = \frac{-H \pm \sqrt{H^2 - 4}}{2}. \tag{8}$$

From this,

$$c = a \ln K. \tag{9}$$

Also from (6),

$$e^{-\frac{b}{a}} = \frac{a}{L - E}\left[-1/K + 1\right]$$

and,

$$e^{\frac{b}{a}} = \frac{L - E}{a}\left[\frac{1}{1 - 1/K}\right]$$

so,

$$b^{3/} = a \ln\left[\frac{L - E}{a}\left(\frac{1}{1 - 1/K}\right)\right].$$

From (8), K can have two possible values; however, in any real situation,

$$K = \frac{-H + \sqrt{H^2 - 4}}{2}. \tag{10}$$

This is because $c > 0$, so $K > 1$ by (9). And because $K^2 + HK + 1 = 0$, the two possible values of K are reciprocals; hence, the larger must be chosen in order for c to be > 0.

3/ The value b may be negative. Negative values result when the low point of the curve is located to the left of the coordinate origin.

In order to change from distances along the tape to true horizontal distances, let z_i be the distance along the tape from $x = 0$, and let x_i be the distance along the x axis.

$$z_i = \int_0^{x_i} \sqrt{1 + \sinh^2 \frac{x_i - b}{a}}\, dx_i$$

$$= a \sinh \frac{x_i - b}{a} - a \sinh \frac{-b}{a}$$

so,

$$\sinh \frac{x_i - b}{a} = \frac{z_i}{a} + \sinh \frac{-b}{a}$$

$$\frac{x_i - b}{a} = \sinh^{-1}\left(\frac{z_i}{a} + \sinh \frac{-b}{a}\right).$$

The accompanying y coordinate (y_i) is

$$y_i = a \cosh \left(\frac{x_i - b}{a}\right).$$

The coordinates of the surface being surveyed are:

$$x_i$$

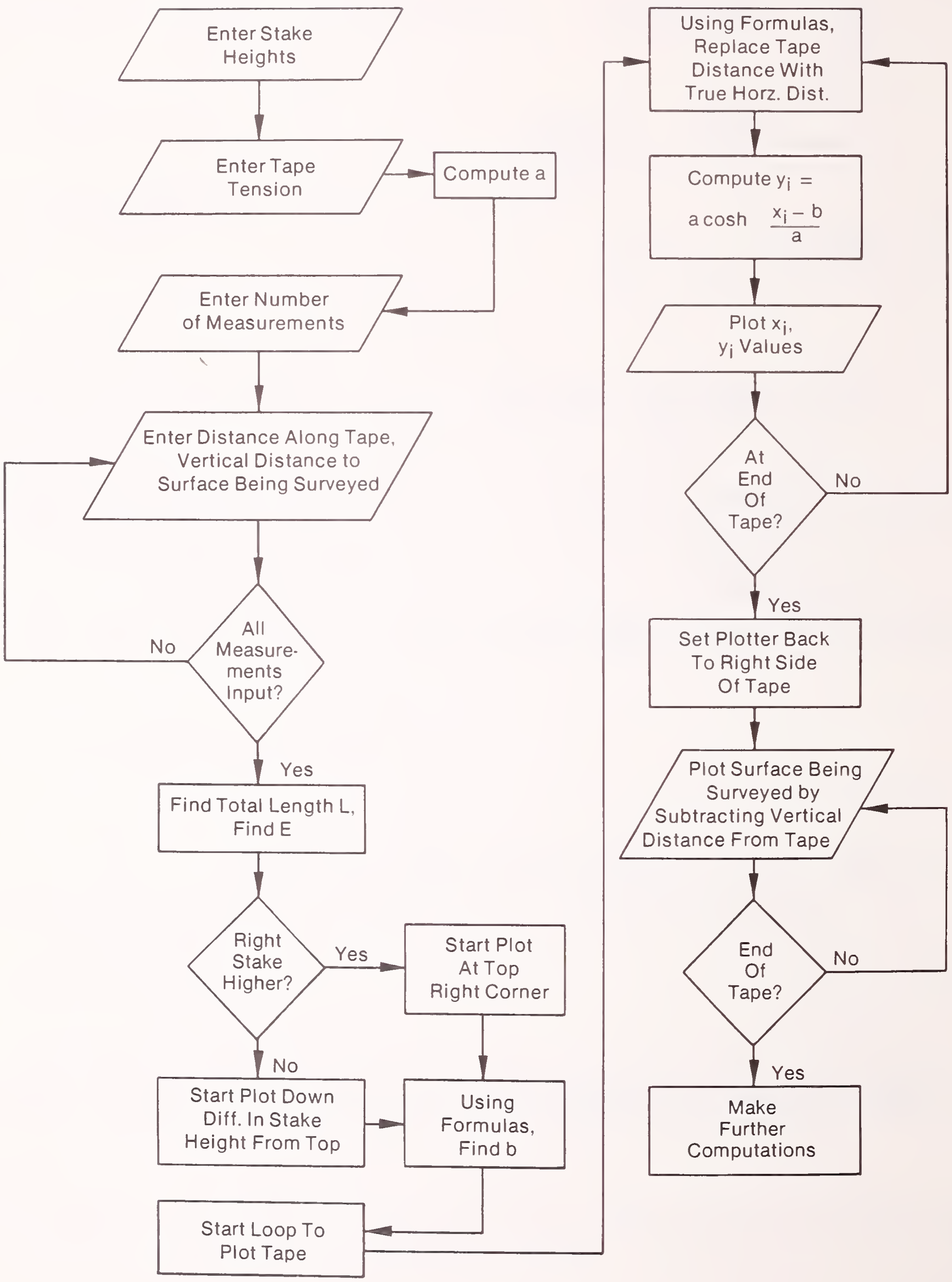
and

$$y_i - d_i$$

where d_i = the vertical distance from the tape to the surface.

APPENDIX 2

Flow chart for calculating and plotting cross sections taken with a sag tape



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KEYWORDS: survey, cross sections, streambank erosion, channel degradation, channel aggradation, aquatic habitat, water profiles, erosion, deposition, profiles.

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